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# NORMALITY OF BLOWING-UP(Related Topics to Cohomologies of Local Rings)

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# NORMALITY OF BLOWING-UP

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## §1. Introduction.

Let  $A$  be a Noetherian local ring with maximal ideal  $m$  and  $d = \dim A > 0$ . Let  $q = (a_1, \dots, a_d)$  be a parameter ideal in  $A$  and put  $R = \bigoplus_{n \geq 0} q^n$ , the Rees ring of  $q$ . In this lecture we shall explore when the scheme  $\text{Proj } R$  is normal and our result is stated as follows:

Theorem(1.1). Suppose that  $\text{depth } A > 0$ . Then the following conditions are equivalent.

(1)  $\text{Proj } R$  is normal.

(2)  $A$  is a regular local ring and  $l_A(q + m^2/m^2) \geq d - 1$ .

When this is the case, the ring  $R$  is a normal ring with divisor class group  $\mathbb{Z}$ . (Here  $l_A(q + m^2/m^2)$  stands for the length of the  $A$ -module  $q + m^2/m^2$ .)

In [6] K. Yamagishi also tackled with this theme and mentioned the equivalence of (1) and (2) in (1.1) with a rather strong assumption that  $A$  is Cohen-Macaulay (cf. [6, Chap. 4, (1.3)]); our theorem guarantees his assumption can be replaced by the weaker

one that  $\text{depth } A > 0$ .

We will prove Theorem(1.1) in the next section. As is noted in (1.1), the ring  $R$  is normal if (and only if)  $\text{Proj } R$  is normal and  $\text{depth } A > 0$ . The normality of  $R$  itself is characterized in divers manners; especially, appealing to a recent result of J. Watanabe [7] on  $m$ -full ideals, we can prove that  $R$  is normal if and only if  $q$  is  $m$ -full. As the fact may have its own significance, in §3 we will discuss this subject a little more closely.

Throughout this lecture let  $A$  denote a Noetherian local ring with maximal ideal  $m$ . We assume that  $\dim A = d > 0$  and fix a parameter ideal  $q = (a_1, \dots, a_d)$  in  $A$ . Let  $R = \bigoplus_{n \geq 0} q^n$  be the Rees ring of  $q$ .

## §2. Proof of Theorem(1.1).

Let  $B = A[x/a_1 \mid x \in q]$  and  $P = mB$ . To begin with we note

Proposition(2.1). (1)  $\dim B = d$ .

(2)  $P$  is a height one prime ideal of  $B$  and  $P = \sqrt{a_1}B$ .

(3) The elements  $a_i/a_1 \bmod P$  ( $2 \leq i \leq d$ ) of  $B/P$  are algebraically independent over  $A/m$ .

Let  $p \in \text{Spec } A$  with  $p \not\ni a_1$ . We put  $I(p) = pA[1/a_1] \cap B$ . Then  $I(p) \in \text{Spec } B$ ,  $I(p) \cap A = p$ , and  $B_{I(p)} = A_p$ . Let  $x^* = x \bmod p$  for each  $x \in A$ .

Lemma(2.2).  $B/I(p) = A/p[x^*/a_1^* \mid x \in q]$  as  $A$ -algebras.

In particular  $\dim B/I(p) \leq \dim A/p$ .

Proof. By definition we get an embedding  $B/I(p) \hookrightarrow A/p[1/a_1^*]$  of  $A$ -algebras, whose image coincides with  $A/p[x^*/a_1^* \mid x \in q]$ .

As  $\dim A/p[x^*/a_1^* \mid x \in q] \leq \dim A/p$  by dimension formula, the second assertion follows from the first.

Corollary(2.3). (1)  $\text{Ass } B = \{ I(p) \mid p \in \text{Ass } A \text{ and } p \not\supset a_1 \}$ .  
 (2)  $\{ I \in \text{Spec } B \mid \dim B/I = d \} = \{ I(p) \mid p \in \text{Spec } A \text{ and } \dim A/p = d \} = \{ I \in \text{Spec } B \mid I \not\subset P \}$ .

Proof. Let  $I \in \text{Spec } B$  and put  $p = I \cap A$ . Then if  $p \not\supset a_1$ , we have  $I = I(p)$  as  $A[1/a_1] = B[1/a_1]$ . Hence we get (1), because  $a_1$  is  $B$ -regular and  $A_p = B_{I(p)}$ . Consider (2). First of all take  $I \in \text{Spec } B$  with  $\dim B/I = d$ . Then as  $\dim B_I = 0$ , we may write  $I = I(p)$  with  $p = I \cap A$ . Notice  $\dim B/I = d \leq \dim A/p$  by (2.2) and we get  $\dim A/p = d$ . Conversely let  $p \in \text{Spec } A$  and assume  $\dim A/p = d$ . Then  $p \not\supset a_1$  clearly. We put  $I = I(p)$ . Recall that  $B/I = A/p[x^*/a_1^* \mid x \in q]$  as  $A$ -algebras and we see by (2.1) that the ring  $B/P+I = (B/I)/m(B/I)$  is a polynomial ring with  $d-1$  variables over the field  $k = A/m$ . Hence the canonical epimorphism  $B/P \rightarrow B/P+I$  of  $k$ -algebras must be an isomorphism, because  $B/P$  and  $B/P+I$  are  $k$ -isomorphic; thus  $P \not\supset I$ . Finally let  $I \in \text{Spec } B$  with  $I \not\subset P$ . Then  $\dim B/I = d$ , as  $\dim B/P = d-1$  — this completes the proof of (2).

Let  $e(A)$  (resp.  $e(B_p)$ ) denote the multiplicity of  $A$  (resp.  $B_p$ ).

Lemma(2.4).  $e(B_p) \geq e(A)$ .

Proof. Let  $h: A[T_2, \dots, T_d] \rightarrow B$  be the  $A$ -algebra map defined by  $h(T_i) = a_i/a_1$  ( $2 \leq i \leq d$ ), where  $T_2, \dots, T_d$  are indeterminates over  $A$ . Let  $K = \text{Ker } h$  and put  $f_i = a_1 T_i - a_i$  ( $2 \leq i \leq d$ ). Then  $K \supset (f_2, \dots, f_d)$ . Notice  $a_1^n K \subset (f_2, \dots, f_d)$  for some integer  $n \geq 1$ , because  $A[1/a_1] = B[1/a_1]$ . Now let  $C = A[T_2, \dots, T_d]_M$

where  $M = mA[T_2, \dots, T_d]$  and consider the exact sequence  $0 \rightarrow L \rightarrow C/(f_2, \dots, f_d)C \rightarrow B_P \rightarrow 0$  of  $C$ -modules. Then as  $a_1^n L = 0$  and as  $a_1, f_2, \dots, f_d$  form a system of parameters for the local ring  $C$ , we have  $l_C(L) < \infty$  and therefore  $e(B_P) = e(C/(f_2, \dots, f_d)C)$ . Recalling that  $e(C/(f_2, \dots, f_d)C) \geq e(C)$ , we get the required inequality  $e(B_P) \geq e(A)$ , as  $e(C) = e(A)$ .

We say that  $A$  is unmixed if  $\dim \hat{A}/p = d$  for any  $p \in \text{Ass } \hat{A}$ , where  $\hat{A}$  denotes the completion of  $A$ . We shall use the following criterion of regularity.

Proposition(2.5)([4,(40.6)]).  $A$  is a regular local ring if and only if  $e(A) = 1$  and  $A$  is unmixed.

The next result (2.6) is a key theorem in this lecture.

Theorem(2.6). Suppose that  $A$  is unmixed. Then the following conditions are equivalent.

- (1)  $A$  is a regular local ring and  $l_A(q + m^2/m^2) \geq d - 1$ .
- (2)  $B_P$  is a DVR.
- (3)  $\text{Proj } R$  is normal.

Proof. (3) $\Rightarrow$ (2) Since  $\text{Spec } B$  appears as one of the affine charts of  $\text{Proj } R$ , this implication is clear.

(2) $\Rightarrow$ (1) As  $e(A) = 1$  by (2.4), we get  $A$  is regular (cf. (2.5)). We will prove that  $l_A(q + m^2/m^2) \geq d - 1$ . Let us maintain the same notation as in Proof of (2.4). Notice that  $K = (f_2, \dots, f_d)$  in our case, since  $a_1, \dots, a_d$  is an  $A$ -regular sequence. Hence  $f_2, \dots, f_d$  is a part of a minimal system of generators for the maximal ideal  $mC$  of  $C$ , because  $B_P = C/(f_2, \dots, f_d)C$  is a DVR by our assumption. Thus  $l_A(q + m^2/m^2) = l_C(qC + m^2C/m^2C) \geq$

$d-1$ , as  $qC = (a_1, f_2, \dots, f_d)C$ .

(1) $\Rightarrow$ (3) As the ring  $R$  is Cohen-Macaulay (cf. [1]), the scheme  $\text{Proj } R$  satisfies the condition  $(S_2)$ . So it is enough to check that all the rings  $A[x/a_i \mid x \in q] \ (1 \leq i \leq d)$  satisfy the condition  $(R_1)$ . We may assume without loss of generality that  $i=1$ . Let  $I \in \text{Spec } B$  with  $\dim B_I = 1$ . Then if  $I \not\ni a_1$ ,  $B_I = A_p$  where  $p = I \cap A$  and  $B_I$  is a DVR in this case. Suppose that  $I \ni a_1$ . Then we get  $I = P$  by (2.1)(2). We must show that  $B_P$  is a DVR. First of all notice that  $m = (a_1, \dots, \hat{a}_1, \dots, a_d, b)$  for some  $1 \leq i \leq d$  and  $b \in m$ , because  $l_A(q + m^2/m^2) \geq d-1$ . We put  $J = bB_P$ . Assume  $i \geq 2$  and write  $a_i = \sum_{j \neq i} a_j x_j$  + by with  $x_j, y \in A$ . Then as  $a_k = a_1 a_k / a_1$  for all  $k$ , we get  $a_1(a_i/a_1 - \sum_{j \neq 1, i} (a_j/a_1)x_j - x_1) \in J$ . Hence  $a_1 \in J$ , because  $a_i/a_1 - \sum_{j \neq 1, i} (a_j/a_1)x_j - x_1 \notin P$  (cf. (2.1)(3)). We can similarly prove that  $a_1 \in J$  for the case  $i=1$  too. Thus  $mB_P = bB_P$ , which guarantees that  $B_P$  is a DVR. This completes the proof of (2.6).

Remark(2.7). Unless  $A$  is unmixed, the implication (2) $\Rightarrow$ (1) in (2.6) is not true in general even though  $A$  is an integral domain and  $B$  is a regular ring. In fact according to M. Nagata [4], there exist a Noetherian local integral domain  $A$  of  $\dim A = 2$  and a system  $a, b$  of parameters for  $A$  which satisfy the following conditions: (1)  $A$  is not a regular ring;

(2)  $B = A[b/a]$  is a regular ring.

Proof. Take a Noetherian local integral domain  $A$  of  $\dim A = 2$  so that (1) the normalization  $\bar{A}$  of  $A$  is a regular ring and only has two maximal ideals, say  $M$  and  $N$ ; (2)  $m = M \cap N$ ; (3)  $\bar{A}$  contains elements  $x$  and  $z$  such that  $M = (x-1, z)\bar{A}$ ,  $N = x\bar{A}$ ,  $z \in N$ , and  $\bar{A} = A + Ax$ . (Such a ring  $A$  must exist, see

[4, p.204].) Then  $A$  is not regular as  $A \neq \bar{A}$ . We put  $a = xz$  and  $b = x(x-1)$ . Then  $a, b$  form a system of parameters in  $A$ . Let us check that  $B = A[b/a]$  is a regular ring. Recall that  $z \in m = M \cap N$ . Then we see  $B \supset \bar{A}$ , as  $B$  contains  $b/a = (x-1)/z$  and as  $\bar{A} = A + A(x-1)$  by (3); hence  $B = \bar{A}[(x-1)/z]$ . Let  $Q$  be a prime ideal of  $B$  and put  $p = Q \cap \bar{A}$ . If  $Q \ni z$ , then  $Q$  contains  $x-1 = z(x-1)/z$  and so we have  $p = M$  by (3). Therefore we get  $B_p = \bar{A}_M[(x-1)/z]$ , which is a regular ring because  $x-1, z$  is a regular system of parameters for  $\bar{A}_M$  (cf. e.g. [2, (4.6)]). Hence the local ring  $B_Q = (B_p)_{Q B_p}$  is regular. If  $Q \not\ni z$ , then  $B_Q = \bar{A}_p$  as  $B[1/z] = \bar{A}[1/z]$  and we have  $B_Q$  is a regular ring also in this case.

Corollary(2.8). The following conditions are equivalent.

- (1)  $B_p$  is a DVR.
- (2) The completion  $\hat{A}$  of  $A$  contains a unique prime ideal  $p$  such that  $\dim \hat{A}/p = d$ . Furthermore  $\hat{A}/p$  is a regular local ring,  $l_{\hat{A}}(q\hat{A} + m^2\hat{A} + p/m^2\hat{A} + p) \geq d-1$ , and  $\hat{A}_p$  is a field.

Proof. We put  $C = \hat{A}[x/a_1 \mid x \in q]$  and  $Q = mC$ . Then  $C$  is a flat extension of  $B$  as  $C = \hat{A} \otimes_A B$ . Notice  $Q = PC$  and  $P = Q \cap B$ . Then we get  $C_Q$  is a DVR if and only if  $B_p$  is; thus we may assume that  $A$  is complete.

(1)  $\Rightarrow$  (2) By (2.4) we get  $e(A) = 1$ . Consequently by the formula  $e(A) = \sum_{p \in \text{Spec } A, \dim A/p = d} l(A_p) \cdot e(A/p)$  (cf. [4, (23.5)]), we find that  $A$  contains a unique prime ideal  $p$  with  $\dim A/p = d$ . Furthermore  $A/p$  is a regular local ring by (2.5), as  $e(A/p) = 1$ . Clearly  $A_p$  is a field. Now we will prove that  $l_A(q + m^2 + p/m^2 + p) \geq d-1$ . Let  $I = I(p)$ . Then by (2.3)(2) we see  $I \not\subset P$ ,

whence  $IB_P = 0$  as  $B_P$  is a DVR. On the other hand we have by (2.2) an isomorphism  $B/I = A/p[x^*/a_1^* \mid x \in q]$  of  $A$ -algebras. Thus the local ring  $(A/p[x^*/a_1^* \mid x \in q])_P$  is a DVR and we conclude by (2.6) that  $l_A(q + m^2 + p/m^2 + p) \geq d - 1$ .

(2)  $\Rightarrow$  (1) Let  $I = I(p)$ . Notice that  $I$  is, by (2.3)(2), a unique prime ideal of  $B$  such that  $I \not\subset P$ . Then we get that  $IB_P = 0$ , as  $B_P$  is a Cohen-Macaulay ring and as  $B_I = A_p$  is a field. Recalling the isomorphism  $B/I = A/p[x^*/a_1^* \mid x \in q]$ , we have that  $B_P = B_P/IB_P$  is a DVR because by (2.6) so is the local ring  $(A/p[x^*/a_1^* \mid x \in q])_P$ .

Corollary(2.9). Assume that  $A$  is a homomorphic image of a Cohen-Macaulay ring and let  $a = a_1$  be a non-zero-divisor of  $A$ . Then the following conditions are equivalent.

(1)  $B$  is a normal ring.

(2) (a)  $A[1/a]$  is a normal ring.

(b)  $A$  contains a unique prime ideal  $p$  such that  $\dim A/p = d$ . Furthermore  $A/p$  is regular and  $l_A(q + m^2 + p/m^2 + p) \geq d - 1$ .

(c) For each  $Q \in \text{Ass } A$  with  $Q \neq p$ , there is an integer  $N \geq 1$  such that  $a^N \in q^{N+1} + Q$ .

Proof. (1)  $\Rightarrow$  (2) As  $A[1/a] = B[1/a]$ , we see  $A[1/a]$  is a normal ring; hence  $A$  is reduced as  $A \subset A[1/a]$ . Notice that  $B$  is integrally closed in the total quotient ring of  $A$ , as it is normal. Then we get by (2.2) and (2.3)(1) an isomorphism  $B = \prod_{Q \in \text{Ass } A} A/Q[x^*/a^* \mid x \in q]$  (#) of  $A$ -algebras. Recall that  $e(A) = 1$  by (2.4) and we find by the formula  $e(A) = \sum_{p \in \text{Spec } A, \dim A/p = d} l(A_p) \cdot e(A/p)$  that  $A$  contains a unique prime ideal  $p$  of  $\dim A/p = d$ . Moreover  $A/p$  is, by (2.5), a regular local ring because  $A/p$  is



unmixed by our standard assumption. Let  $Q \in \text{Ass } A$  such that  $Q \neq p$ . Then we get by the isomorphism (#) that  $A/Q[x^*/a^* \mid x \in q] = m \cdot (A/Q[x^*/a^* \mid x \in q])$ , since  $P = mB$  is a prime ideal of  $B$  and since  $A/p[x^*/a^* \mid x \in q] \neq m \cdot (A/p[x^*/a^* \mid x \in q])$  by (2.1)(2). Hence we find that  $B_p = (A/p[x^*/a^* \mid x \in q])_p$  is a DVR and that the element  $a^* = a \bmod Q$  is invertible in the ring  $A/Q[x^*/a^* \mid x \in q]$ . Thus  $l_A(q + m^2 + p/m^2 + p) \geq d-1$  by (2.6) and  $a^N \in q^{N+1} + Q$  for some  $N \geq 1$ .

(2) $\Rightarrow$ (1) Let  $J \in \text{Spec } B$  and we will show that the local ring  $B_J$  is normal. If  $J \not\supset a$ , this follows from (a) because  $B[1/a] = A[1/a]$ . Assume  $J \supset a$ , or equivalently  $J \supset P$ .

Claim.  $J \not\supset I(Q)$  for any  $Q \in \text{Ass } A$  such that  $Q \neq p$ .

For, suppose that  $J \supset I(Q)$  for some  $Q \in \text{Ass } A$  with  $Q \neq p$ . Then as  $a^* = a \bmod Q$  is invertible in the ring  $A/Q[x^*/a^* \mid x \in q]$  (cf. (c)), we find by (2.2) that  $I(Q) + P = B$  whence  $J = B$  — this is a contradiction.

By this claim and the embedding  $B \subset \prod_{Q \in \text{Ass } A} B/I(Q)$  (recall that  $\bigcap_{Q \in \text{Ass } A} I(Q) = 0$  in  $B$ , see (2.3)(1)), we get that the ring  $B_J$  appears as a local ring of  $C = A/p[x^*/a^* \mid x \in q]$ . Hence  $B_J$  is normal by (2.6).

Example(2.10). Let  $S = k[[X, Y, Z, W]]$  be a formal power series ring over a field  $k$  and let  $I = (X) \cap (Y, Z) \cap (X-Y, Z, W)$  in  $S$ . We put  $A = S/I$ ,  $a = Z^2 - X^2 \bmod I$ ,  $b = Y - X \bmod I$ , and  $c = W - X \bmod I$ . Then  $q = (a, b, c)$  is a parameter ideal in  $A$  and  $B = A[b/a, c/a]$  is a normal ring.

Proof. To check that  $q$  is a parameter ideal in  $A$  is routine. To see that  $B$  is normal, let  $x = X \bmod I$ ,  $y = Y \bmod I$  and  $z = Z \bmod I$ . Then  $m = (x, b, c, z)$ ; hence  $b, c$  form a

part of a regular system of parameters for the ring  $A/xA$ . As  $a \in (q^2 + (y, z)) \cap (q^2 + (x-y, z, w))$  and as  $A[1/a]$  is normal, our assertion follows from (2.9).

Lemma(2.11). Let  $I = (b_1, \dots, b_s)$  be an  $m$ -primary ideal of  $A$ . Then  $A[x/b_i \mid x \in I] \neq m \cdot A[x/b_i \mid x \in I]$  for some  $1 \leq i \leq s$ .

Proof. Assume the contrary and take an integer  $N \geq 1$  so that  $b_i^N \in mI^N$  for all  $i$ . Let  $G = \bigoplus_{n \geq 0} I^n / I^{n+1}$  and put  $f_i = b_i \bmod I^2$ . Then as  $f_i^N \in mG$ , we find that all the  $f_i$ 's are nilpotent in  $G$ , whence  $d = \dim G = 0$  — this is a contradiction.

In the situation of (2.11) we don't have always  $A[x/b_i \mid x \in I] \neq m \cdot A[x/b_i \mid x \in I]$ . (For instance, consider  $A = k[[t^2, t^3]]$  and  $I = (t^2, t^3)$ .) This is, of course, the case when  $b_1, \dots, b_s$  is a system of parameters in  $A$ , cf. (2.1).

We now prove Theorem(1.1).

Proof of Theorem(1.1). (2)  $\Rightarrow$  (1) See (2.6).

(1)  $\Rightarrow$  (2) According to (2.6) we have only to show that  $A$  is unmixed. We put, as in Proof of (2.8),  $C = \hat{A}[x/a_1 \mid x \in q]$  and  $Q = mC$ . Let  $N$  be a maximal ideal of  $C$  such that  $N \supset Q$ .

Claim 1.  $\dim C_N / QC_N = d - 1$ .

Proof. The ideal  $N/Q$  is maximal in the ring  $C/Q$  and so we have that  $\dim C_N / QC_N = d - 1$ , since  $C/Q$  is a polynomial ring with  $d - 1$  variables over the field  $A/m$ , cf. (2.1)(3).

Claim 2.  $\dim C_N / I = d$  for any  $I \in \text{Ass } C_N$ .

Proof. Notice that  $\text{Ass}_B B/a_1 B = \{P\}$  as  $B$  is normal. Then we have  $\text{Ass}_C C/a_1 C = \{Q\}$  as  $B/a_1 B \cong C/a_1 C$ . Let  $I \in \text{Ass } C_N$  and take  $J \in \text{Ass}_{C_N} C_N/a_1 C_N$  so that  $J \supset I$  (this choice is possible as  $a_1$  is  $C_N$ -regular, cf. e.g. [3, (15.D)]). Then we must have, as

$\text{Ass}_{C_N} C_N/a_1 C_N = \{QC_N\}$ , that  $J = QC_N$  whence  $\dim C_N/J = \dim C_N/QC_N = d-1$  by Claim 1. Thus  $\dim C_N/I = d$  since  $J \not\supseteq I$ .

Let us check that  $A$  is unmixed. Assume the contrary and pick  $p \in \text{Ass } \hat{A}$  so that  $\dim \hat{A}/p < d$ . Then we get by (2.11)  $\hat{A}/p[x^*/a_i^* \mid x \in q] \neq m \cdot (\hat{A}/p[x^*/a_i^* \mid x \in q])$  (#) for some  $1 \leq i \leq d$ , where  $x^* = x \bmod p$  for each  $x \in \hat{A}$ . We may assume  $i=1$ . Recall that the ideal  $q$  is generated by non-zero-divisors of  $A$ , because  $\text{depth } A > 0$  by our standard assumption. Hence we may further assume that  $a = a_1$  is a non-zero-divisor of  $A$ . Then as  $p \nmid a$ , we get by (2.2) an isomorphism  $C/I = \hat{A}/p[x^*/a^* \mid x \in q]$  of  $\hat{A}$ -algebras, where  $I = I(p)$ . According to (#) this isomorphism guarantees that  $Q+I \neq C$ , whence we may choose a maximal ideal  $N$  of  $C$  so that  $N \supset Q+I$ . Then as  $I \in \text{Ass } C$  by (2.3)(1), we get  $\dim C_N/IC_N = d$  by Claim 2 — this is quite impossible since by (2.2)  $\dim C_N/IC_N \leq \dim C/I \leq \dim \hat{A}/p < d$ . Thus  $A$  is unmixed. The proof of the last assertion of (1.1) shall be given in the next section, see (3.1).

Remark(2.12).  $\text{Proj } R$  is not necessarily regular even though  $\text{Proj } R$  is normal and  $\text{depth } A > 0$ . In fact, provided  $d \geq 2$  and  $\text{depth } A > 0$ ,  $\text{Proj } R$  is regular if and only if  $A$  is a regular local ring and  $q = m$  (cf. [2, (4.6)]).

### §3. Normality of the ring $R$ .

In this section we discuss the normality of the ring  $R = \bigoplus_{n \geq 0} q^n$  and our goal is

Theorem(3.1). The following conditions are equivalent.

- (1)  $A$  is regular and  $l_A(q + m^2/m^2) \geq d-1$ .

- (2)  $A$  is an integral domain and  $q$  is integrally closed.
- (3)  $q$  is  $m$ -full.
- (4)  $R$  is normal.

When this is the case, the divisor class group  $C(R)$  of  $R$  is an infinite cyclic group.

To begin with we recall the definition of  $m$ -full ideals.

Let  $I$  be an ideal of  $A$ . Then we say that  $I$  is  $m$ -full if  $mI : x = I$  for some  $x \in m$ .

The concept of  $m$ -full ideals was introduced by D. Rees [5] and basic properties of such ideals are discussed in [7], a few of which we need to prove (3.1).

Let  $v_A(M)$  denote the number of elements in a minimal system of generators for a finitely generated  $A$ -module  $M$ .

Proposition(3.2)([7, Theorem 2 and 3]). Let  $I$  be an  $m$ -primary ideal of  $A$  and assume that  $I$  is  $m$ -full. Let  $x \in m$  such that  $mI : x = I$ . Then  $v_A(J) \leq v_A(I) = l_A(A/I + xA) + v_A(I + xA/xA)$  for any ideal  $J$  of  $A$  containing  $I$ .

Let  $I$  be an ideal of  $A$ . Then an element  $x$  of  $A$  is called integral over  $I$  if  $x$  satisfies an equation  $x^N + c_1 x^{N-1} + \dots + c_N = 0$  with  $c_i \in I^i$  ( $1 \leq i \leq N$ ). Recall that  $I$  is said to be integrally closed if every element of  $A$  which is integral over  $I$  belongs to  $I$ .

The next result is due to D. Rees and a proof may be found in [7] (cf. Theorem 5).

Proposition(3.3). Suppose that  $A$  is an integral domain with

infinite residue class field. Then every integrally closed ideal of  $A$  is  $m$ -full.

Proof of Theorem(3.1). (4) $\Rightarrow$ (2) Let  $N = mR + R_+$  and  $P \in \text{Ass } R$ . Then  $P \subset N$ , as  $P$  is graded and as  $N$  is a unique graded maximal ideal of  $R$ . As  $R_N$  is normal, it is an integral domain and so  $PR_N = 0$ , whence  $P = 0$ . Thus  $R$  is an integral domain and so  $A$  is. Let us identify  $R$  with the  $A$ -subalgebra  $A[cT \mid c \in q]$  of  $A[T]$  where  $T$  is an indeterminate over  $A$ . Let  $c \in A$  which is integral over  $q$ . Then as  $cT$  is integral over  $R$ , we get  $cT \in R$ ; hence  $cT \in qT$ , that is  $c \in q$ . Thus  $q$  is integrally closed.

(3) $\Rightarrow$ (1) Take  $x \in m$  so that  $mq : x = q$ . Then by (3.2) we find that  $v_A(m) \leq v_A(q) = l_A(A/q + xA) + v_A(q + xA/xA)$ . So  $A$  is a regular local ring, since  $v_A(m) \leq v_A(q) = d$ . Furthermore we get  $l_A(A/q + xA) = 1$ , because  $l_A(A/q + xA) \geq 1$  and  $v_A(q + xA/xA) \geq d - 1$ . Thus  $q + xA = m$ , that is  $l_A(q + m^2/m^2) \geq d - 1$ .

(2) $\Rightarrow$ (1) Passing to the ring  $A[U]_{mA[U]}$  where  $U$  is an indeterminate over  $A$ , we may assume that the field  $A/m$  is infinite. Then as  $q$  is  $m$ -full by (3.3), our implication follows from (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3) Let  $x \in m$  with  $m = (a_1, \dots, \hat{a}_i, \dots, a_d, x)$  for some  $1 \leq i \leq d$ . Then we get  $l_A(A/q + xA) + v_A(q + xA/xA) = l_A(A/m) + v_A(m/xA) = d$  (a). Recalling the exact sequence  $0 \rightarrow mq : x/mq \rightarrow A/mq \xrightarrow{x} A/mq \rightarrow A/mq + xA \rightarrow 0$  of  $A$ -modules, we have  $l_A(mq : x/mq) = l_A(A/mq + xA)$  (b). Notice that  $l_A(A/q + xA) + v_A(q + xA/xA) = l_A(A/q + xA) + l_A(q + xA/mq + xA) = l_A(A/mq + xA)$ . Then we get by (a) and (b) that  $l_A(mq : x/mq) = l_A(q/mq)$  (c), as  $l_A(q/mq) = v_A(q) = d$ .

Since  $mq : x \supset q$ , it follows from (c) that  $mq : x = q$ . Thus  $q$  is  $m$ -full.

(1) $\Rightarrow$ (4) Let  $N = mR + R_+$ . Then as  $\text{Proj } R$  is normal (cf. (2.6)), we get that the local ring  $R_P$  is normal for any prime ideal  $P$  ( $P \neq N$ ) of  $R$ . On the other hand as  $R$  is a Cohen-Macaulay ring (cf. [1]), we see  $\text{depth } R_N = \dim R_N = d+1 \geq 2$ ; so the local ring  $R_N$  must be normal too. This completes the proof of the equivalence of the conditions in (3.1).

Let us compute the divisor class group  $C(R)$  of  $R$ . We may assume  $d \geq 2$ . Let  $e = e_q(A)$  and choose a minimal system  $b_1, \dots, b_d$  of generators for  $m$  so that  $q = (b_1^e, b_2, \dots, b_d)$ . We put  $p = b_1 A$ ,  $P = pA[T] \cap R$ , and  $Q = mR$ . Then we have

- Claim. (1)  $P$  and  $Q$  are height one prime ideals in  $R$ .  
 (2)  $P \cap A = p$  and  $Q \cap A = m$ .  
 (3)  $b_1 R = P \cap Q$ .

Proof. (1) As  $R/P \cong R_{A/p}(q+p/p)$ , we get  $\dim R/P = d$ ; hence  $\dim R_P = 1$ . Recall that  $R \cong A[T_1, \dots, T_d]/I$  as  $A$ -algebras, where  $C = A[T_1, \dots, T_d]$  is a polynomial ring over  $A$  and  $I$  denotes the ideal of  $C$  generated by all the  $2 \times 2$  minors of the matrix  $\begin{pmatrix} T_1 & T_2 & \cdots & T_d \\ b_1^e & b_2 & \cdots & b_d \end{pmatrix}$ . Hence  $R/Q \cong A/m[T_1, \dots, T_d]$  and we get  $Q = mR$  is a height one prime ideal of  $R$ .

(2) This is clear.

(3) It suffices to show that  $\text{Ass}_R R/b_1 R = \{P, Q\}$ ,  $b_1 R_P = PR_P$ , and  $b_1 R_Q = QR_Q$ . As  $P \cap Q \ni b_1$ ,  $\text{Ass}_R R/b_1 R \supset \{P, Q\}$  clearly. Let  $P' \in \text{Ass}_R R/b_1 R$  and put  $p' = P' \cap A$ . Notice  $\dim R_{P'} = 1$  since  $R$  is normal. If  $p' = m$ , then  $P' \supset Q$ ; hence  $P' = Q$ . Assume  $p' \neq m$ . Then  $R_{p'} = A_{p'}[T]$  and  $P'R_{p'} \supset p'R_{p'} \neq 0$ .

Consequently  $P'R_{p'} = p'R_{p'}$  (as  $\dim R_{p'} = 1$ ) and so we find  $\dim A_{p'} = 1$ . Thus  $p' = p$  (recall  $p' \ni b_1$ ) and therefore  $P'R_p = pR_p$ . Because  $P \in \text{Ass}_R R/b_1R$  and  $P \cap A = p$ , we see  $P'R_p = pR_p$  whence  $p' = P$ . Thus  $\text{Ass}_R R/b_1R = \{P, Q\}$ . Furthermore as  $pR_p = pR_p (= b_1R_p)$ , we find  $b_1R_p = pR_p$ . Let  $\tilde{R} = R[1/b_1^e T]$  and we get  $Q\tilde{R} = b_1R$  (recall  $b_i = b_1^e b_i^e T / b_1^e T$  for each  $2 \leq i \leq d$ ). Therefore  $b_1R_Q = QR_Q$  as  $Q \not\ni b_1^e T$ , which completes the proof of Claim.

By this claim and the fact that  $R[1/b_1] = A[1/b_1][T]$  is a UFD, we see that  $C(R)$  is generated by  $\text{cl}(Q) (= -\text{cl}(P))$ . We must show that the order of  $\text{cl}(Q)$  is not finite. Assume the contrary and choose an integer  $n > 0$  so that  $n \cdot \text{cl}(Q) = 0$ . Then  $Q^{(n)} = bR$  for some  $b \in R$  and, as  $b_1^n \in Q^{(n)}$ , we may write  $b_1^n = bc$  with  $c \in R$ . Notice that  $b, c \in A = R_0$ , because  $b_1^n \neq 0$  and  $R$  is a graded integral domain. Moreover we find  $c \notin Q$  since  $b_1^n R_Q = Q^n R_Q = bR_Q$ . Therefore  $c$  is a unit of  $A$  and we get  $Q^{(n)} = bR = b_1^n R$ ; consequently  $P \supset Q$  (as  $P \ni b_1$ ). This is of course impossible and we conclude that  $C(R) = \mathbb{Z}$  as required.

Remark(3.4). Let  $B_i = A[x/a_i \mid x \in q]$  for  $1 \leq i \leq d$ . We put  $e = e_q(A)$  and assume  $R$  is normal. Then we get for each  $1 \leq i \leq d$ , similarly as in Proof of (3.1), that the divisor class group  $C(B_i)$  of  $B_i$  is a finite cyclic group and  $|C(B_i)| \mid e$ . The proof is not complicated which we leave to readers.

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Added in proof. After giving this lecture, the author was told that a similar result as the equivalence of the conditions (1) and (2) in Theorem(3.1) had been obtained also by D. Katz (A criterion for complete-intersections to be self-radical, Arch. Math., 42 (1984), 423 - 425). However according to our Theorem(1.1), his main result Theorem 1 is not correct and therefore the proof of Corollary 6 and 7 in the paper is not complete. The author guarantees that they are immediate consequences of our Theorems (1.1) and (3.1).